

LEFSCHETZ TYPE PENCILS ON CONTACT MANIFOLDS

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ABSTRACT

We define the concept of Lefschetz contact pencil and we show the existence of such structures on any contact manifold. The main idea of the proof is a generalization of the Donaldson arguments used in the symplectic case. We will analyze some of the applications of such existence theorem for the topology of approximately holomorphic contact submanifolds. In particular, we study the topological relationship between the contact submanifolds constructed in [IMP99].

1. INTRODUCTION.

S. Donaldson in [Do96] has adapted the concept of very ample bundle to the symplectic setting. Following this idea some results of complex projective geometry has been generalized to symplectic geometry, as Bertini's theorem and Lefschetz's hyperplane theorem in [Do96], connectedness of the space of "good" sections of a very ample bundle in [Au97], divisors on projective fibrations in [Pa98], special position theorems in [Pa99], existence of Lefschetz pencils in [Do99], existence of branched coverings of symplectic 4-manifolds over \mathbb{CP}^2 and associated invariants [Au99, Au99b], and Kodaira's embeddings theorems and symplectic determinantal submanifolds in [MPS99]. These ideas have opened a new insight in symplectic geometry allowing to understand symplectic manifolds through the study of the linear systems associated to a "very ample" vector bundle.

In [IMP99] the idea of [Do96] of working with approximately J -holomorphic sections was partially translated to the contact case. With these sections and a generalization of the local estimated transversality result of [Do96], which is the key of the symplectic approach, the ideas of [Do96, Au97] were translated word by word to the contact setting. The only important loss was the isotopy results which have been developed in the symplectic theory starting with the ideas of D. Auroux in [Au97]. In this paper we show how to develop a contact geometry of linear systems analogous to the symplectic case. We also show how to recover partially the isotopy in the theory.

We will prove a theorem analogue to that of [Do99]. In fact, we will show the existence of a certain class of pencils on a contact manifold. The main tool in the proof will be a generalization of the local transversality theorem proved in [Do99]. As in the symplectic case this result is not strictly necessary for the proof, but it has interest in its own. It could be used to simplify the constructions of other linear systems, as for instance, the one in [MPS99] in the symplectic case.

A compatible chart in a contact manifold (C, D) at a point x will be a chart $\phi : U_x \subset C \rightarrow \mathbb{C}^n \times \mathbb{R}$, where U_x is neighborhood of x , verifying $\phi(x) = (0, 0)$, $(\phi_*)(D(x)) = \mathbb{C}^n \times \{0\}$ and moreover verifying that the presymplectic form $(\phi_*)d\theta(x)$, when restricted to $\mathbb{C}^n \times \{0\}$, is a positive form of type $(1, 1)$ at the origin of coordinates. If the contact manifold is exact we will impose also that $\langle \phi_*(R)(x), \frac{\partial}{\partial s} \rangle > 0$ and say that the chart is oriented compatible, where R is the Reeb vector field and s is the real coordinate.

Definition 1.1. *A (oriented) contact pencil on a closed (exact) contact manifold C consists of the following data:*

1. *a codimension 4 contact submanifold $A \subset C$,*
2. *a finite set of smooth contact curves $\Delta = \bigcup_{i \in I} \gamma_i \subset C - A$,*
3. *a smooth map $f : V - A \rightarrow \mathbb{CP}^1$, whose restriction to the complementary of the set Δ is a submersion, satisfying also that $P_\Delta =$*

$f(\Delta)$ is a set of locally smooth curves with transversal selfintersections.

Also the data have to admit the following standard local models:

- At any point $a \in A$, there are (oriented) compatible coordinates $(z_1, \dots, z_n, s) \in \mathbb{C}^n \times \mathbb{R}$ such that A is locally given by $\{z_1, z_2 = 0\}$. And the function f has the expression $f(z_1, \dots, z_n, s) = \frac{z_1}{z_2} \in \mathbb{CP}^1$ near a .
- At a point $b_i \in \gamma_i$ there are (oriented) compatible coordinates in which f is written as $f(b_i) + \varphi(s) + z_1^2 + \dots + z_n^2$, where $\varphi : \mathbb{R} \rightarrow \mathbb{C}$ verifies $\varphi(0) = 0$ and $\varphi'(0) \neq 0$.

It is clear from the local model that the counterimage $f^{-1}(p)$ is a subset of $C - A$, whose closure in C is smooth at $f^{-1}(p) \cap A$. Abusing language, we will call fiber over b to the closure of the counterimage. It is a smooth submanifold if b is a regular value of f . In other case we will have one or two singularities locally modelled by:

$$(1) \quad \varphi(s) + z_1^2 + \dots + z_n^2.$$

In case $\dim C = 3$, the smooth fibers will be (oriented) links on C . The link operation that is performed when the image crosses a circle γ_i of the sphere \mathbb{CP}^1 looks, after general projection to a plane, as

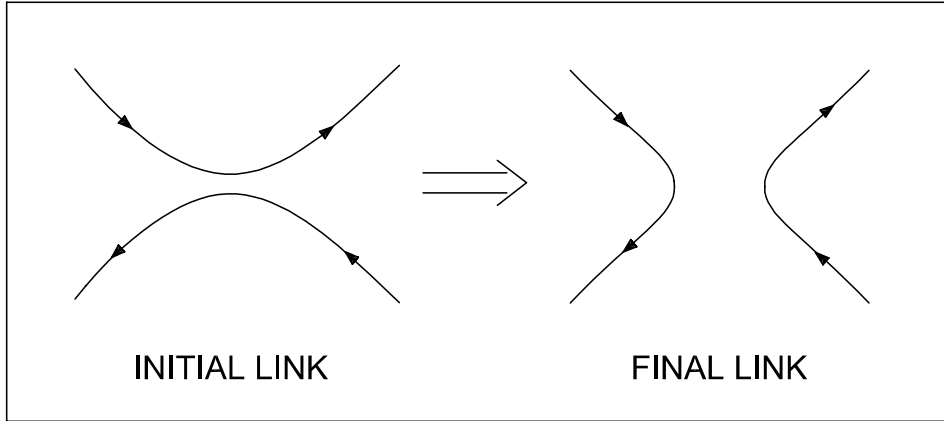


FIGURE 1. *Link operation*

The main result of this article is

Theorem 1.2. *Given a contact closed manifold (C, D) (resp. exact) and $\alpha \in H_{2n-1}(C, \mathbb{R})$ which is reduction of an integer class, there exists a contact pencil on C (resp. oriented) whose fibers are contact submanifolds, homologous to α .*

With this result at hand, it is easy to understand the possibility of finding general isotopic constructions for the contact submanifolds constructed in [IMP99]. Once fixed a compatible complex structure in the distribution, in Section 6 we will show how to define a sequence of contact fibrations f_k , satisfying $f_k(0) \simeq N_k$ and $f_k(\infty) \simeq N'_k$, where

N_k and N'_k are sequences of codimension 2 contact submanifolds constructed with the method developed in [IMP99]. In order to assure the isotopy between N_k and N'_k we have only to construct a path between 0 and ∞ in \mathbb{CP}^1 which does not intersect Δ , because in this case f restricted to the path is surjective. But this is only possible if 0 and ∞ are in the same connected component of $\mathbb{CP}^1 - \Delta$, which is not true in general (contrarely to the symplectic case where Δ consists of isolated points). We will study in Subsection 6.1 the topological relationship between the counterimages of the points of a path crossing Δ . This will prove that

$$\begin{aligned} H_i(N_k) &= H_i(N'_k), \\ \pi_i(N_k) &= \pi_i(N'_k), \end{aligned}$$

for $k = 0, \dots, n-2$. Moreover we will show how to make the result independent of the chosen complex structure and of the contact form, thus providing a comprobaton of the “contact Lefschetz hyperplane theorem” proved in [IMP99]. The result brings us back to the Lefschetz original ideas to prove the hyperplane theorem (see [Le24]). It would be interesting to complete the proof in Lefschetz’s way, since this would be a more geometrical proof than the one contained in [IMP99], the latter being an adaptation of the modern Morse-theoretic argument due to Bott, Andreotti and Frankel (and in the symplectic case to Donaldson).

In Section 2 we will give the basic results in contact geometry needed to develop the proof. In Section 3 we will give the proof of the exact case assuming transversality results of approximately holomorphic contact geometry. Afterwards, in Sections 4 and 5, we develop the local approximately holomorphic techniques needed to achieve this transversality. In Section 6 we will study the relationship between approximately holomorphic sections transverse to $\mathbf{0}$. Finally, we will adapt all the study to the non-exact case in Section 7.

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2. DEFINITIONS AND RESULTS.

We will assume along the proofs that (C, D) is an exact contact manifold, where we have fixed a global contact form θ . In Section 7

¹Geometría simpléctica con técnicas algebraicas.

we will precise the changes needed to extend the results to the non-exact setting. We start recalling with the basic definitions and results of [IMP99].

2.1. Basic concepts. A complex structure on C is a complex structure J defined on D , interpreted as a symplectic vector bundle, compatible with the symplectic form $d\theta$. Recall that the distribution D is contact, if and only if the restriction of $d\theta$ to D is symplectic. The contact 1-form defines a vector field R by the condition

$$i_R\theta = 1, \quad i_Rd\theta = 0,$$

It is called the Reeb vector field. Given a complex structure we obtain a metric for the manifold

$$g_J(u, v) = d\theta(u, Jv) + \theta \otimes \theta.$$

This metric is called a contact metric. It depends on the fixed contact form. The k -rescaled contact metric will be defined as $g_{J,k} = kg_J$. If there is not risk of confusion we will denote simply g_k , supposing J fixed along the proofs. (Observe that the k -rescaled contact metric g_k is not the contact metric associated to $k^\alpha\theta$, for $\alpha \in \mathbb{R}$)

Now we give some definitions to control contact structures in \mathbb{R}^{2n+1} .

Definition 2.1. *The maximum angle between two subspaces $U, V \in \text{Gr}_{\mathbb{R}}(r, n)$ is defined as:*

$$\angle_M(U, V) = \max_{u \in U} \angle(u, V).$$

This angle defines a distance in the topological space $\text{Gr}_{\mathbb{R}}(r, n)$ (for details see [MPS99]).

Definition 2.2. *Let θ_k be a sequence of contact forms in \mathbb{R}^{2n+1} , with associated distributions $D_k = \text{Ker } \theta_k$. The sequence is called c -asymptotically flat in the set $U \subset \mathbb{R}^{2n+1}$ if*

$$\angle_M(D_k(0), D_k(x)) \leq ck^{-1/2}, \quad \text{for all } x \in U.$$

The sequence is called asymptotically flat if there exists some $c > 0$ for which it is c -asymptotically flat.

The standard contact structure in \mathbb{R}^{2n+1} is defined as $\theta_0 = ds + \sum_{j=1}^n x_j dy_j$, where $(x_j, y_j, s) \in \mathbb{R}^{2n+1}$. The k -rescaled contact metric is the contact metric associated to $\theta_{k^{1/2}} = k^{1/2}ds + \sum x_j dy_j$, which is obtained from θ_0 scaling the coordinates by a factor $k^{1/2}$. It is obvious from the definition that $\theta_{k^{1/2}}$ is a sequence of asymptotically flat contact forms on any bounded set of \mathbb{R}^{2n+1} .

If a contact distribution D is defined at the origin by the horizontal subspace $\mathbb{R}^{2n} \times \{0\} \subset \mathbb{R}^{2n+1}$, we can define at a neighborhood of this point a canonical complex structure in D by means of vertical projection of the canonical one defined on \mathbb{R}^{2n} . This complex structure will be denoted J_0 . In the case of the standard contact structure we can extend J_0 over all \mathbb{R}^{2n+1} .

Following [IMP99] we can define ∂ and $\bar{\partial}$ operators in any kind of function, morphism or section defined on a contact manifold, restricting ourselves to the contact distribution, or equivalently projecting along the Reeb direction.

2.2. Approximately holomorphic geometry. We will talk about uniform constant, polynomial, when this constant, polynomial, etc. does not depend on the chosen point $x \in C$, nor in the integer k appearing in the context. (However these constants can depend on the modulus of the given sections, on the nature of its derivatives, on the step of a recurrent reasoning, etc. but always independently of k).

We will use from [IMP99] the following

Lemma 2.3. *Given (C, θ) a closed contact manifold and J a compatible complex structure on $\text{Ker } \theta$, there exists a uniform constant $c > 0$ and a contact Darboux chart $\psi : (C, \theta) \rightarrow (\mathbb{R}^{2n+1}, \theta_0)$ satisfying*

$$\frac{1}{2}g(v, w) \leq \langle (\psi_*)_x v, (\psi_*)_x w \rangle \leq 2g(v, w), \forall x \in B_g(x, c), v, w \in T_x C.$$

This implies that $|\nabla^r \psi| = O(1)$ and $|\nabla^r \psi^{-1}| = O(1)$, for $r = 1, 2, 3$. Moreover $|\bar{\partial}\psi(y)| \leq c'd(x, y)$, for a uniform constant c' .

A $k^{1/2}$ -Darboux chart is a chart $\phi_k : B(x, \epsilon) \rightarrow \mathbb{R}^{2n+1}$ such that $(\phi_k)_* \theta = \theta_{k^{1/2}}$. The following result is a direct corollary of Lemma 2.3:

Corollary 2.4. *Given (C, θ) a closed contact manifold and J a compatible complex structure on $\text{Ker } \theta$, there exists a uniform constant $c > 0$ such that there exists a contact chart $\psi : (C, \theta) \rightarrow (\mathbb{R}^{2n+1}, \theta_{k^{1/2}})$ satisfying*

$$\frac{1}{2}d_{g_k}(v, w) \leq \langle (\psi_*)_x v, (\psi_*)_x w \rangle \leq 2d_{g_k}(v, w), \forall x \in B_{g_k}(x, c), v, w \in T_x C.$$

This implies that $|\nabla^r \psi| = O(1)$ and $|\nabla^r \psi^{-1}| = O(1)$, for $r = 1, 2, 3$. Moreover $|\nabla^r \bar{\partial}\psi(y)| \leq c'k^{-1/2}$, for $r = 0, 1, 2$, for a uniform constant c' .

These results allows to trivialize locally contact manifolds in an approximately holomorphic way. The analogous notion of the c -bounds of [Do99] in the contact case is the following

Definition 2.5. *A sequence of sections s_k of hermitian bundles E_k over the contact manifold (C, θ) has mixed C^r -bounds (c_D, c_R) at the point $x \in C$ if it satisfies*

$$\begin{aligned} |s_k(x)| &< c_D, \\ |\nabla_D^j s_k(x)| &< c_D, \quad \forall j = 1, \dots, r. \\ |\nabla^j s_k(x)| &< c_R, \quad \forall j = 1, \dots, r. \\ |\nabla^j \bar{\partial} s_k(x)| &< c_R k^{-1/2}, \quad \forall j = 0, \dots, r-1. \end{aligned}$$

The sequence has uniform mixed C^r -bounds (c_D, c_R) if it satisfies these bounds at every point.

There we denote by ∇_D the restriction of the operator ∇ to the subspace D . The metric used in the manifold C to measure the norms in the precedent definition is the rescaled contact metric g_k . Recall that if s_k^j has (c_D^j, c_R^j) mixed C^r -bounds ($j = 1, 2$), then $s_k^1 + s_k^2$ has $(c_D^1 + c_D^2, c_R^1 + c_R^2)$ mixed C^r -bounds.

The following results are used to trivialize bundles over contact manifolds.

Definition 2.6. *A sequence of sections s_k of bundles E_k has mixed Gaussian decay in C^r -norm away from a point $x \in C$ if there exist a uniform polynomial P and uniform constants $\lambda > 0$, c_a such that for all $y \in X$, the sequence $s_k(y)$ has mixed C^r -bounds*

$$(P(d_k(x, y)) \exp(-\lambda d_k(x, y)^2), c_a P(d_k(x, y)) \exp(-\lambda d_k(x, y)^2)).$$

Recall that the uniform constant c_a is necessary because we will check afterwards that for different sequences of sections this constant cannot be fixed.

The prequantizable line bundle L over an exact contact manifold C is defined as the complex line bundle with connection such that $\text{curv}(L) = id\theta$.

Lemma 2.7 (Lemma 5 from [IMP99]). *Let (C, θ) be a closed contact manifold. There exists a uniform constant $c_s > 0$, such that given any point $x \in C$, there exists a sequence of sections $\sigma_{k,x}$ of $L^{\otimes k}$ satisfying $|\sigma_{k,x}| \geq c_s$ at every y in a ball of g_k -radius 10 centered at x and the sections $\sigma_{k,x}$ have uniform mixed Gaussian decay away from x in C^3 -norm (in this case $c_a = 1$).*

2.3. Transversality results. Following Donaldson [Do99] and Auroux [Au99] we set up the following definitions. A linear application $f : \mathbb{R}^n \rightarrow \mathbb{R}^r$ is η -transverse if it has a right inverse $\nu : \mathbb{R}^r \rightarrow \mathbb{R}^n$ such that $|\nu| \leq \eta^{-1}$. In the non-linear case we will say that $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^r$ is η -transverse to $y \in \mathbb{R}^r$ over U if $\forall x \in U$, such that $|f(x) - y| < \eta$, then df is η -transverse. Recall that this is an open condition. In fact, if $|f - g|_{C^1, U} < \epsilon/10$ and f is ϵ -transverse to y over U then g is, say, $\epsilon/2$ -transverse to y in U .

The definition of transversality to $\mathbf{0}$ for sections of hermitian bundles over riemannian manifolds is totally analogous. In the case of contact manifolds we have to strengthen the conditions.

Definition 2.8. *A section s_k of the hermitian vector bundle E_k over the contact manifold (C, θ) is η -transverse to $\mathbf{0}$ on $U \subset C$ if for all $x \in U$ such that $|s_k(x)| < \eta$ then $\nabla_D s_k(x)$ is η -transverse to $\mathbf{0}$ (with respect to the g_k -metric in C).*

From the discussion of [IMP99] it follows that a sequence of sections s_k of the bundles E_k over the contact manifold (C, θ) , which has uniform (c_D, c_R) bounds and which is η -transverse to $\mathbf{0}$, has as zero set a contact submanifold, for k large enough. The precise result of [IMP99] is

Theorem 2.9. *Given a closed exact contact manifold (C, θ) . Let $\epsilon > 0$ and let s_k be a sequence of sections of the bundles $E \otimes L^{\otimes k}$, for a fixed hermitian bundle E , with uniform mixed C^r -bounds (c_D, c_R) . Then there exists a real number $\eta > 0$ (depending on ϵ , c_D and c_R), and a sequence σ_k such that:*

1. $\sigma_k - s_k$ has mixed C^r -bounds (ϵ, c'_R) .
2. σ_k is η -transverse to $\mathbf{0}$.

In [IMP99] the proof is developed for sequences with mixed C^2 -bounds, but there is not any problem in generalizing it to the mixed C^r -bounds case. We need in this article mixed C^3 bounds. Given a section $s_k = (s_k^0, s_k^1)$ of the bundle $\mathbb{C}^2 \otimes S \otimes L^{\otimes k}$, for a fixed hermitian line bundle S , whose zero set is $Z(s_k)$, we denote $F^{s_k} = \frac{s_k^1}{s_k^0} : C - Z(s_k) \rightarrow \mathbb{CP}^1$ the projectivization of the section. The holomorphic part of the differential of this application will be denoted by ∂F^{s_k} . Now we state a generalization of Theorem 2.9 which will be proved in Section 4:

Theorem 2.10. *Given a closed exact contact manifold (C, θ) . Let $\epsilon > 0$ and let $s_k = (s_k^0, s_k^1)$ be a sequence of sections of the bundles $\mathbb{C}^2 \otimes S \otimes L^{\otimes k}$ with mixed C^3 -bounds (c_D, c_R) . Suppose that s_k^0 and s_k are both transverse to $\mathbf{0}$. Then there exists $\eta > 0$ (depending on ϵ , c_D and c_R), and a sequence σ_k satisfying*

1. $\sigma_k - s_k$ has mixed C^3 -bounds (ϵ, c'_R) ,
2. σ_k^0 and $\sigma_k^0 \oplus \sigma_k^1$ are η -transverse to $\mathbf{0}$,
3. ∂F_k^σ is η -transverse to $\mathbf{0}$ away from $Z(\sigma_k^0)$.

The techniques used in Section 4 improve slightly the ones of [IMP99] and thus could allow a simpler proof of Theorem 2.9 avoiding some of the complications of the globalization process in that article.

3. PROOF OF THE MAIN RESULT.

Take a complex line bundle S with connection ∇ satisfying that $\text{curv}(\nabla) = PD(\alpha)$. This is possible since α is an integer class. Starting with any sequence of sections s'_k of $\mathbb{C}^2 \otimes S \otimes L^{\otimes k}$ with mixed C^3 -bounds, we can perturb it using Theorems 2.9 and 2.10 to achieve a sequence of sections s_k verifying properties 2 and 3 of Theorem 2.10 and with mixed C^3 -bounds (c_D, c_R) . We consider this sequence as starting datum and will use it to construct the oriented contact pencil.

From the η -transversality of s_k the zero set $A = Z(s_k)$ is a codimension 4 contact manifold where F^{s_k} is not well defined. We will write F instead of F^{s_k} whenever it causes no confusion. Now we will study, as in [Do99], the shape of the “bad set” $\Gamma = \{x \in C : |\partial F| \leq |\bar{\partial} F|\}$. From [IMP99] we know that if we prove that $\Gamma = \{x \in C : d_D F = 0\}$ we will have obtained that the fibres of F are contact at all smooth points. This will be proved in several steps.

Lemma 3.1. *There is a constant $\xi > 0$, depending only on c_D , c_R and ϵ , such that if k is large enough then $|s_0| \geq \xi$ on Γ .*

Proof: The proof is analogous to that of Lemma 7 in [Do99]. \square

Define Δ as the set of critical points of F . The connected components of Δ form a discrete set of smooth curves by the transversality condition imposed to ∂F . Also we can assure that this set of components is finite because it is contained in Γ , which by Lemma 3.1 is contained in the complementary of a γ -neighborhood of $W_\infty = Z(s_k^0)$, for $\gamma > 0$ a uniform constant small enough. We define:

$$\Omega_\xi = \{p \in C, |s_0(p)| > \xi/2\}.$$

The following step, adapting again Donaldson's argument, is to estimate the shape of the set Δ . This is the content of the following:

Proposition 3.2. *There is a uniform constant $\rho_0 > 0$, such that the ρ_0 -neighborhoods of each connected component γ_i of Δ are disjoint and are contained in Ω_ξ . Moreover for any $\rho < \rho_0$, for k large enough (only depending on ρ), the set Γ is contained in a ρ -neighborhood of the set Δ .*

The proof of this result is absolutely analogous to the proof of Proposition 9 in [Do99] and depends strongly on Lemma 8 in that paper. We refer the reader to [Do99] for the argument.

Finally, we need to perturb the sequence ∂F^{s_k} in arbitrarily small neighborhoods of A and Δ to achieve the local models required in Definition 1.1. The perturbation required in Δ needs a careful analysis, but again the situation in A is a straightforward generalization of [Do99]. We need only to define

$$LD_x = \nabla s_0^k \oplus \nabla s_1^k : TC_x \rightarrow L_x^{\otimes k} \oplus L_x^{\otimes k}.$$

With this notation the result we need in our case is

Lemma 3.3. *For a point $x \in A$, F can be represented in the standard model of Definition 1.1 at x if and only if $TA_x \cap D$ is a symplectic subspace and the restriction of $d\theta$ to the symplectic orthogonal CA_x (in D) of $TA_x \cap D$ is a positive form of type $(1,1)$ with respect to the complex structure on CA_x induced by LD_x .*

We do not provide a proof of this Lemma, which follows word by word the proof of Lemma 11 in [Do99]. Thanks to the (c'_D, c'_R) mixed C^3 -bounds and the transversality of the sequence s_k , it is easy to check that a small C^3 -perturbation of the sequence satisfies the hypothesis of Lemma 3.3, thus completing the study in the neighborhood of A .

Now we study the map F near Δ . Again Donaldson's ideas work in this case, however the adaptation of the proof needs some changes. Select a smooth connected curve γ_i in Δ . We are going to perturb F in a γ -neighborhood of Δ . By Lemma 3.2, the perturbations can be made in each connected component γ_i in an independent way.

Recall that, for k large enough, the curve γ_i is contact, i.e. $\forall x \in \gamma_i$, $T_x C = T_x \gamma_i \oplus D_x$. Moreover, the angle between $T_x \gamma_i$ and D_x is bounded below by a uniform constant because of the transversality of the sequence. Using the contact metric g_J associated to the fixed complex structure J to define a geodesic flow, we can obtain a diffeomorphism:

$$\phi_i : U_\rho \rightarrow V_\rho \subset S^1 \times \mathbb{C}^n,$$

where U_ρ is the ρ -neighborhood of γ_i (in g_k metric) and V_ρ is its image by the flow, which is an open neighborhood of $S^1 \times \{0\}$. We can construct a metric in S^1 by imposing the condition that $(\phi_i)|_{\gamma_i}$ is an isometry with respect to the rescaled contact metric g_k . In \mathbb{C}^n we will fix the standard metric. The product metric will be denoted by g_0^k . We can select with this choice a uniform $\rho > 0$ such that

$$(2)$$

$$\lambda_m g_k(d\phi_i(v), d\phi_i(w)) \leq |g_0^k(v, w)| \leq \lambda_M g_k(d\phi_i(v), d\phi_i(w)), \quad \forall v, w \in T_x C, \forall x \in U_\rho,$$

where $\lambda_m, \lambda_M > 0$ are uniform constants. Once we have fixed ϕ_i , we obtain a distribution D_k in $\phi_i(U_\rho)$ constructed as the image of the distribution D . Denote by D_h the integrable distribution given as $\{p\} \times \mathbb{C}^n$ defined in $S^1 \times \mathbb{C}^n$. Perhaps after shrinking ρ uniformly, we can check that

$$(3) \quad \angle_M(D_k(s, z), D_h(s, z)) < c_u |z| k^{-1/2}, \quad \forall (s, z) \in V_\rho \subset S^1 \times \mathbb{C}^n,$$

where $c_u > 0$ is a uniform constant. Moreover we can impose without loss of generality that $(\phi_i)_* J(\gamma_i) = J_0$. So we can project orthogonally $(\phi_i)_* J$ (defined on D_k) to D_h obtaining a new almost complex structure \hat{J} . In fact it is easy to check, as in [Do96], that

$$(4) \quad \partial = \partial_0 + \bar{\mu} \bar{\partial}_0, \quad \bar{\partial} = \bar{\partial}_0 + \mu \partial_0,$$

where ∂ and ∂_0 are the operators defined by the structures \hat{J} and J_0 in $S^1 \times \mathbb{C}^n$, and $|\mu(z)| \leq c|z|k^{-1/2}$, where c is a uniform constant. In order to finish the proof we follow these steps: we will define the perturbation, afterwards we will prove it satisfies the conditions for the distribution D_h with the almost complex structure \hat{J} , and finally we will check the result for the distribution D_k .

Given any differentiable function $f : C \rightarrow \mathbb{C}$, we denote $f_0 = f \circ \phi_i^{-1}$. By the inequalities (2) we can use F_0 instead of F for all the computations using the induced distribution D_k . To construct the perturbation we define the complex Hessian $H = \frac{1}{2} \partial \bar{\partial} F$. Using the trivialization ϕ_i we may regard it as

$$H_0(s, z) = \sum H_{\alpha\beta}(s) z_\alpha z_\beta,$$

on $\phi_i(U_\rho)$. Also we take a cut-off function $\beta_\rho : S^1 \times \mathbb{C}^n \rightarrow [0, 1]$ satisfying

1. $\beta_\rho(\phi_i(p)) = 1$, if $d_k(p, \gamma_i) \leq \frac{\rho}{2}$.
2. $\beta_\rho(\phi_i(p)) = 0$, if $d_k(p, \gamma_i) \geq \rho$.

$$3. |\nabla\beta_\rho| = O(\rho^{-1}).$$

We can adjust β to assure condition 3 because of equation (2). The constant $\rho < \rho_0$ will be fixed along the proof to assure that the conditions are satisfied (namely we will have to shrink ρ in a uniform way). A modification of F will be

$$f_0(s, z) = \beta_\rho(w'(s) + H_0(s, z)) + (1 - \beta_\rho)F_0(s, z),$$

where $w' : S^1 \rightarrow \mathbb{C}$ is any smooth function.

We denote by ∂ and $\bar{\partial}$ the operators associated to the almost complex structure \tilde{J} acting on D_h . Respectively we denote by ∂_k and $\bar{\partial}_k$ the operators associated to the action of $(\phi_i)_*J$ in D_k . The equivalent to Lemma 10 of [Do99] is

Lemma 3.4. *If $\rho > 0$ is small enough, k is sufficiently large, $|w'(s) - F_0(s, 0)|$ is sufficiently small and $|\frac{dw'(s)}{ds}| = O(1)$, then the inequality $|\partial_k f_0| \leq |\bar{\partial}_k f_0|$ is only satisfied in γ_i .*

Proof: First, assume that we are at a point where $\beta_\rho = 1$. Then $f_0 = w' + H_0$ and

$$\partial f_0 = \partial H_0, \quad \bar{\partial} f_0 = \bar{\partial} H_0.$$

The η -transversality of ∂F_0 yields the bound

$$|\partial H_0(s, z)| \geq \eta|z| - |\bar{\partial}(\partial F_0)_{(z=0)}||z|.$$

Now recall that $\bar{\partial}\partial + \partial\bar{\partial} = 0$ on functions and that the norm of $\partial\bar{\partial}F_0$ is controled because F has uniform mixed C^3 -bounds. Then we can write

$$|\partial H_0(s, z)| \geq \eta|z| - c_u k^{-1/2}|z|.$$

Using the inequality (3) and that $|\frac{\partial}{\partial s} H_{\alpha\beta}(s)| = O(1)$ (as follows from the mixed C^3 -bounds of F) and recalling the bound $|\frac{dw'}{ds}| = O(1)$, we rewrite the inequality as

$$(5) \quad |\partial_k H_0(s, z)| \geq \eta|z| - c'_u k^{-1/2}|z|,$$

where $c'_u > 0$ is a uniform constant. On the other hand,

$$|\bar{\partial} H_0| \leq c_u |z|^2 k^{-1/2},$$

and thus, by an analogous argument,

$$(6) \quad |\bar{\partial}_k H_0| \leq c''_u |z|^2 k^{-1/2} + c''_u |z| k^{-1/2}.$$

Now, if we impose that $|\partial_k H_0| \leq |\bar{\partial}_k H_0|$, we obtain by comparing (5) and (6), $z = 0$ for k large enough.

Now we study points in the annulus containing the support of $\nabla\beta_\rho$. In this case,

$$\bar{\partial} f_0 = \bar{\partial}\beta_\rho(w' + H_0 - F_0) + \beta_\rho \bar{\partial} H_0 + (1 - \beta_\rho) \bar{\partial} F_0.$$

Bounding the right hand side as in [Do99] we obtain an expression for the value $|\bar{\partial}f_0|$. Using (3) and the bounds $|\frac{dw'}{ds}| = O(1)$ and $|\bar{\partial}_k F_0| = O(k^{-1/2})$, we conclude

$$|\bar{\partial}_k f_0| \leq c(\rho^2 + k^{-1/2} + |F_0(s, 0) - w'| \rho^{-1}).$$

In the same way we know that

$$\partial_k f_0 = \partial_k \beta_\rho(w' + H_0 - F_0) + \beta_\rho \partial_k H_0 + (1 - \beta_\rho) \partial_k F_0.$$

Using the transversality of F we can obtain a lower bound for $|\partial_k f_0|$. The argument follows the one of Donaldson arriving to the final expression

$$|\partial_k f_0| - |\bar{\partial}_k f_0| \geq \frac{\eta\rho}{2} - c(\rho^2 + k^{-1/2} + |w' - F_0(s, 0)| \rho^{-1}).$$

Obviously, once fixed a sufficiently small ρ , for k large enough and $|F_0(s, 0) - w'|$ small enough compared with ρ the inequality is strictly positive for any point in the annulus. \square

To finish the proof we only have to check that the function f conforms the local models at any point of Δ . We use the perturbation w' to assure that the curves γ_i project smoothly into \mathbb{CP}^1 , this is equivalent to impose

$$\nabla_v f(x) \neq 0,$$

for any point $x \in \Delta$ and any nonzero $v \in T_x \Delta$. This can be achieved by a generic perturbation, and so w' can be selected to get it. Also we can assure, using this perturbation, that the intersections of branches of $f(\Delta)$ are transverse, again by a genericity argument.

Finally, given any $x \in \Delta$ there exists coordinates $(s, x_1, y_1, \dots, x_n, y_n)$ such that f is locally written as

$$f(s, z_1, \dots, z_n) = \varphi(s) + \sum H_{\alpha\beta}(s) z_\alpha z_\beta,$$

where $\varphi'(0) \neq 0$ (this condition is equivalent to the smoothness of the branches of $f(\Delta)$). Now at a neighborhood of $s = 0$ the 1-parametric family $(H_{\alpha\beta}(s))$ of bilinear complex forms can be diagonalized by a smooth family of complex changes of coordinates if the eigenvalues of $(H_{\alpha\beta})$ are all distinct. This is a genericity condition that can be achieved at all the points of Δ by a generic perturbation of $O(k^{-1/2})$ in F , before starting the perturbation process which we have developed along this section. With this condition, we obtain a smooth family of invertible complex matrix $P(s)$ such that

$$\sum H_{\alpha\beta}(s) z_\alpha z_\beta = z^T P(s)^T P(s) z,$$

where $z = (z_1, \dots, z_n)$. Therefore the change of coordinates

$$\begin{aligned} \mathbb{R} \times \mathbb{C}^n &\rightarrow \mathbb{R} \times \mathbb{C}^n \\ (s, z) &\rightarrow (s, P(s)z) \end{aligned}$$

gives us the required local model. This finishes the proof of the main theorem.

4. TRANSVERSALITY RESULTS.

In this section we will prove Theorem 2.10.

4.1. The globalization scheme. First we recall how the globalization process developed in [Do96] adapts to the contact setting. This adaptation was carried out in [IMP99]. Now, we set up the process in a functorial way in the style of [Au99]. As always, we denote by (C, θ) an exact contact manifold.

Definition 4.1. *A family of properties $\mathcal{P}(\epsilon, x)_{x \in C, \epsilon > 0}$ of sections of bundles over C is local and mixed C^r -open if, given a section s satisfying $\mathcal{P}(\epsilon, x)$, any section σ such that $s - \sigma$ has (η, c_R) mixed C^r -bounds satisfies $\mathcal{P}(\epsilon - c_u \eta, x)$, for some constant c_u .*

Proposition 4.2 ([IMP99]). *Let $\mathcal{P}(\epsilon, x)_{x \in C, \epsilon > 0}$ be a local and mixed C^r -open family of properties of sections of vector bundles E_k over C . Assume there exist uniform constants c, c', c'', p and a function $f : \mathbb{R}^3 \rightarrow \mathbb{R}^+$ such that, given any $x \in C$, any small enough $\delta > 0$, and mixed C^r -bounded sections s_k of E_k with uniform mixed C^r -bounds, say (c_D, c_R) , there exist, for all large enough k , mixed C^r -bounded sections $\tau_{k,x}$ of E_k with the following properties:*

1. $\tau_{k,x}$ has mixed C^r -bounds $(c''\delta, f(c''\delta, c_D, c_R))$,
2. the sections $\frac{1}{\delta}\tau_{k,x}$ have mixed Gaussian decay away from x in C^r -norm,
3. $s_k + \tau_{k,x}$ satisfy the property $\mathcal{P}(\eta, y)$ for all $y \in B_{g_k}(x, c)$, with $\eta = c'\delta(\log(\delta^{-1}))^{-p}$.

Then, given any $\alpha > 0$ and mixed C^r -bounded sections s_k of E_k , there exist, for all large enough k , mixed C^r -bounded sections σ_k of E_k , such that $s_k - \sigma_k$ has mixed C^r -bounds (α, c_R) for some $c_R > 0$. Also, the sections σ_k satisfy $\mathcal{P}(\epsilon, x)$ for some uniform $\epsilon > 0$ at any $x \in C$.

Sketch of the proof: Although this is just a slight variation of the globalization argument in [IMP99] we provide the main lines by completeness. Let S be a finite set of points in C verifying the following properties:

1. $\bigcup_{x \in S} B_{g_k}(x, c) \supset C$.
2. There exists a partition $S = \bigcup_{j \in J} S_j$ verifying that $d_{g_k}(x, y) > \mathcal{N}$ if $x, y \in S_j$. \mathcal{N} will be fixed along the proof.
3. The cardinal of J is $O(\mathcal{N}^{2n+1})$.

The idea is to achieve the property $\mathcal{P}(\epsilon, x)$ in all the balls $B_{g_k}(x, c)$, $x \in S_j$ at once. Using the hypothesis at each $x \in S_j$ we can build a local section $\tau_{k,x}$ which satisfies $\mathcal{P}(\eta, y)$ for all $y \in B_{g_k}(x, c)$, where $\eta = c''\delta(\log(\delta^{-1}))^{-p}$. We select \mathcal{N} large enough, such that at a given $x \in S_j$ the components of the perturbations due to the rest of the points of the set S_j through the directions defined by the distribution D do

not destroy the transversality obtained at $B_{g_k}(x, c)$. The condition we impose to \mathcal{N} is

$$(7) \quad c''\delta(\log(\delta^{-1}))^{-p} \geq 2c_u c' \delta \exp(-\lambda \mathcal{N}^2),$$

where c_u is a uniform constant. The left hand side of the inequality is the amount of transversality obtained by the perturbation at a point $x \in S_j$. The right hand side is a value less than the double of the value of the norm of the sum of all the other perturbations in the set S_j (different from the selected and multiplied by the uniform constant provided by the local an C^r -open property). So the inequality assures us that, after adding the perturbation, we have obtained $\mathcal{P}(\frac{1}{2}c''\delta(\log(\delta^{-1}))^{-p}, y)$ at a c -neighborhood of S_j .

Now the process is clear. We perturb in S_1 to obtain $\mathcal{P}(\eta_1, x)$ at a c -neighborhood of it. Afterwards we add again a perturbation with mixed C^r -bounds $(\frac{1}{c_u}\eta_1/2, c_R^2)$ to achieve $\mathcal{P}(\min(\eta_2, \eta_1/2), y)$ for all y at a c -neighborhood of $S_1 \cup S_2$. The number of steps is independent of k , so the achieved final property is $\mathcal{P}(\eta, x)$ where η does not depend on k . The added section has also mixed C^r -bounds (ϵ, c_R) . The constant ϵ is obtained by adjusting the η_i in the iteration, which is possible by property 1. The constant c_R is uniform because at each stage does depend only on the precedent values of c_R and η_j , so it is independent of k , and thus the uniformity is obvious since the number of stages is independent of k (i.e. $O(\mathcal{N}^{2n+1})$). Remark that c_R cannot be bounded since property 1 provides uniformity but not control of the constant!

To end we have to check the inequality (7) in all the steps of the process. Following the asymptotic analysis of [Do96] we conclude that it is possible to obtain an integer \mathcal{N} independently of the step j . This ends the proof. \square

4.2. The local perturbation. We are going to achieve the transversality property for ∂F , required in Theorem 2.10, by using Proposition 4.2. The first observation is the following

Lemma 4.3. *There exists a uniform constant $\xi > 0$, such that, for k large enough, any point x verifying $|\partial F(x)| < \xi$ lies in the set $\Omega_{2\xi}$.*

The proof is identical to that of Lemma 3.1 and we refer again to [Do99] for details.

We are going to perturb the sequence of sections s_k^1 on order to obtain transversality. The open mixed C^2 -open property $\mathcal{P}(\epsilon, x)$ which we have to obtain in C is that ∂F is ϵ -transverse to $\mathbf{0}$ at x . We are in the hypothesis of Definition 4.1. Now we are going to construct a local section satisfying the hypothesis of Proposition 4.2, thus concluding the proof. We can choose $c > 0$ uniformly to assure that $B_{g_k}(x, 8c) \subset \Omega_\xi$ for any $x \in \Omega_{\frac{3}{2}\xi}$ and also that $B_{g_k}(x, 8c)$ is in the complementary set of Ω_ξ for any x in the complementary set of $\Omega_{\frac{3}{2}\xi}$.

If we are in the complementary set of $\Omega_{\frac{3}{2}\xi}$ we can choose the section $\tau_{k,x}$ to be zero, because of Lemma 4.3.

Choose a point $x \in \Omega_{\frac{3}{2}\xi}$. Take an approximately contact holomorphic chart (s, z_k^1, \dots, z_k^n) satisfying the properties of Corollary 2.4. Generalizing [Au99, MPS99] we define the 1-forms

$$\mu_k^j = \partial\left(\frac{z_k^j s_{k,x}^{ref}}{s_k^0}\right).$$

In Ω_ξ these forms have uniform mixed C^3 -bounds because the term $\nabla^r \bar{\partial} z_k^j$ can be bounded near x , and furthermore both $s_{k,x}^{ref}$ and s_k^0 have mixed C^3 -bounds, as well as s_k^0 is also bounded below. Recall that $(\mu_k^j)_{j=1,\dots,n}$ is a unitary basis of D^* at x , and it is almost unitary in the ball $B_{g_k}(x, 8c)$ (i.e. the basis μ_k^j is arbitrarily close to a unitary basis), after eventually shrinking c uniformly. We define an application $v : B_{g_k}(x, 8c) \rightarrow \mathbb{C}^n$ by the formula:

$$\partial F = \sum_{j=1}^n v_j \mu_k^j.$$

In matrix notation,

$$(8) \quad \partial F = v^T \cdot \mu_k,$$

where $\mu_k^T = (\mu_k^1, \dots, \mu_k^n)$. Thus it is possible to understand μ_k as a linear map $\mu_k(y) : \mathbb{C}^n \rightarrow T_y^* M$. Multiplying by μ_k^{-1} in (8) we obtain

$$v^T = \partial F \cdot \mu_k^{-1}.$$

This implies that $|v| = O(1)$, since μ_k is approximately unitary. We compute now the derivatives of v using equation (8). To do this, recall that ∂F has mixed C^2 -bounds in the ball $B_{g_k}(x, 8c)$ (because F has mixed C^3 -bounds). Also, in the same way, μ_k has mixed C^2 -bounds. Differentiating (8)

$$\nabla \partial F = \nabla v^T \cdot \mu_k + v^T \cdot \nabla \mu_k,$$

which implies that $|\nabla v^T \cdot \mu_k| = O(1)$ and using again that μ_k is approximately unitary, we finally get

$$|\nabla v| = O(1)$$

Differentiating respect to $\bar{\partial}$ we find that $|\bar{\partial} v| = O(k^{-1/2})$, and iterating the process

$$|\nabla \nabla v| = O(1), \quad |\nabla \bar{\partial} v| = O(k^{-1/2}).$$

Now we use the approximately contact-holomorphic chart Ψ defined in Corollary 2.4, after eventual uniform shrinking of c . We construct the function $\hat{v} = v \circ \Psi^{-1}$. By the properties of Ψ , it is easy to check that:

$$|\hat{v}| = O(1), \quad |\nabla^r \hat{v}| = O(1), \quad |\nabla^{r-1} \bar{\partial} \hat{v}| = O(k^{-1/2}), \quad r = 1, 2.$$

Scaling the coordinates by a uniform constant we can assume that $\Psi(B_{g_k}(x, 2c)) \subset B_{2n+1}(0, 2) \subset \Psi(B_{g_k}(x, 8c))$. Now, we are in the hypothesis of the following

Proposition 4.4. *Let $f_k: B \times [0, 1] \rightarrow \mathbb{C}^m$ be a sequence of functions where B is the ball of radius 1 in \mathbb{C}^n and $B \times [0, 1]$ is equipped with a sequence of contact forms $\theta(k)$ whose distributions are asymptotically flat. Let $0 < \delta < 1/2$ be a constant and let $\sigma = \delta(\log(\delta^{-1}))^{-p}$, where p is a integer depending only on the dimensions. Assume that f_k satisfies over $B \times [0, 1]$ the following bounds*

$$|f_k| \leq 1, \quad |\bar{\partial}_0 f_k| \leq \sigma, \quad |\nabla \bar{\partial}_0 f_k| \leq \sigma,$$

for k large enough, where $\bar{\partial}_0$ is the $(0, 1)$ operator defined in $D(k) = \ker \theta(k)$ by vertical projection of the standard complex structure J_0 . Then for k large enough there exists a smooth curve $w_k: [0, 1] \rightarrow \mathbb{C}^m$ such that $|w_k| < \delta$ and the function $f_k - w_k$ is σ -transverse to zero on $B(0, 1/2) \times [0, 1]$. Moreover, if $|\partial f_k / \partial s| < 1$ and $|\partial \nabla f_k / \partial s| < 1$, we can choose w_k such that $|d^i w_k / ds^i| < \Phi(\delta)$, $(i = 1, 2)$; $d^j w_k / ds^j(0) = 0$ and $d^j w_k / ds^j(1) = 0$, for all $j \in \mathbb{N}$, where c is a uniform constant and $\Phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a function depending only on the dimensions.

This proposition will be proved in Section 5. This is the analogous in the contact case to Theorem 12 in [Do99]. For the particular value of $m = 1$ it has been proved in [IMP99].

Now we apply this proposition to the map \hat{v} over $[-1, 1] \times B$, because for k large enough, it satisfies the hypothesis (without loss of generality we can choose $[-1, 1] \times B$ instead of $[0, 1] \times B$, also we suppose $|\hat{v}|_{C^2} \leq 1$, multiplying by uniform constants). The obtained path \hat{w} is extended to $\mathbb{R} \times \mathbb{C}^n$ as

$$\hat{w}(s, z) = \begin{cases} \hat{w}(1) & , \text{for } s > 1, \\ \hat{w}(s) & , \text{for } s \in [-1, 1], \\ \hat{w}(-1) & , \text{for } s < -1. \end{cases}$$

We keep the same notation for this extended map. The map $\hat{v} - \hat{w}$ is transverse to $\mathbf{0}$ in $B(0, 1) \times [-1, 1]$, thus $\hat{v} - \hat{w}$ is also transverse to $\mathbf{0}$ in $B_{2n+1}(0, 1)$. Remark that the chart Ψ is defined in a ball of radius $O(k^{1/2})$. So the pull-back of \hat{w} by Ψ , denoted w , is well defined in this ball, and it will be enough for our purposes. By the properties of Ψ we have that $v + w$ is η -transverse to $\mathbf{0}$ over the ball $B_{g_k}(x, c)$. The constant η does not coincide with the transversality obtained by $\hat{v} + \hat{w}$ by a uniform factor, so $\eta = c_u \delta(\log(\delta^{-1}))^{-p}$, the constants c_u and p are uniform and δ is the norm of w except by a uniform factor $c_{u'}$, i.e. $|w| < c_{u'} \delta$, $c_{u'} \neq 0$.

The needed perturbation is

$$\tau_{k,x} = \sum_{j=1}^n w_j z_j s_{k,x}^{ref}.$$

The first question is whether $\tau_{k,x}$ has the desired mixed C^3 -bounds. The verification is mere routine and it follows the lines of [IMP99]. As an example, we compute the bound for $\bar{\partial}\tau_{k,x}$:

$$\bar{\partial}\tau_{k,x} = \sum \bar{\partial}w_j z_j s_{k,x}^{ref} + w_j \bar{\partial}z_j s_{k,x}^{ref} + w_j z_j \bar{\partial}s_{k,x}^{ref}.$$

The third term is easily bounded by

$$\begin{aligned} |w_j z_j \bar{\partial}s_{k,x}^{ref}| &\leq O(1)2d_k(x,y)c_a k^{-1/2}P(d_k(x,y))\exp(-\lambda d_k(x,y)^2) = \\ &= k^{-1/2}Q(d_k(x,y))\exp(-\lambda d_k(x,y)^2). \end{aligned}$$

The second one is bounded in the same way recalling that $\bar{\partial}z = c_0 k^{-1/2}d_k(x,y)$, where $c_0 > 0$ is uniform. For the first term we proceed as follows:

$$|\bar{\partial}w_j z_j s_{k,x}^{ref}| \leq |\bar{\partial}w_j|P(d_k(x,y))\exp(-\lambda d_k(x,y)^2).$$

Recall that $\theta_{k^{1/2}}$ is asymptotically flat. Using this fact we obtain that $|\nabla w_j| \leq O(k^{-1/2})d_k(x,y)$, thus in particular $|\bar{\partial}w_j| \leq O(k^{-1/2})d_k(x,y)$. This concludes the bounding.

Now we have to compare $\tau_{k,x}$ with w . If $\partial \frac{\tau_{k,x}}{s_k^0}$ were $w^T \mu_k$, the proof would be rapidly concluded. But we are almost in this situation because

$$\partial \frac{\tau_{k,x}}{s_k^0} = \Sigma \partial w_j \frac{z_j s_{k,x}^{ref}}{s_k^0} + \Sigma w_j \mu_j.$$

Using (3) again, we obtain

$$\partial \frac{\tau_{k,x}}{s_k^0} = O(k^{-1/2}) + w^T \mu_k.$$

Therefore, we can find \tilde{w} such that $\partial \frac{\tau_{k,x}}{s_k^0} = \tilde{w}^T \mu_k$ and $|w - \tilde{w}| = O(k^{-1/2})$. So, for k large enough $v - \tilde{w}$ is, say, 0.9η -transverse to $\mathbf{0}$ in $B_{g_k}(x, c)$. We end by remarking that $\partial \frac{s_k^{1-\tau_{k,x}}}{s_k^0}$ is $c_b \eta$ -transverse to $\mathbf{0}$, $c_b > 0$ a uniform constant, if $v - \tilde{w}$ is 0.9η -transverse to $\mathbf{0}$. This is obvious since both just differ by the application of the almost unitary matrix μ_k . \square

5. LOCAL RESULTS.

The aim of this section is to prove Theorem 4.4. This is the generalization of the local results in the symplectic setting, needed to achieve controlled transversality in the contact case.

5.1. Reduction to integral distributions. We can easily reduce the proof to the following

Proposition 5.1. *Let $f: B \times [0, 1] \rightarrow \mathbb{C}^m$ be a complex valued function, where B is the ball of radius 1 in \mathbb{C}^n . Let $0 < \delta < 1/2$ be a constant and let $\sigma = \delta(\log(\delta^{-1}))^{-p}$, where p is a suitable fixed integer depending only on the dimensions n, m . Assume that f_s satisfies the following bounds over $B \times [0, 1]$*

$$|f_s| \leq 1, \quad |\bar{\partial}f_s| \leq \sigma, \quad |\nabla \bar{\partial}f_s| \leq \sigma.$$

Then there exists a smooth curve $w: [0, 1] \rightarrow \mathbb{C}^m$ such that $|w| < \delta$ and the function $f_s - w(s)$ is σ -transverse to zero over the ball $B(0, 1/2)$. Moreover, if $|\partial f_s / \partial s| < 1$ and $|\partial \nabla f_s / \partial s| < 1$, we can choose w such that $|d^i w / ds^i| < \Phi(\delta)$ ($i = 1, 2$), $d^j w / ds^j(0) = 0$ and $d^j w / ds^j(1) = 0$ for all $j \in \mathbb{N}$, where Φ is a function depending only on the dimensions n, m .

The proof of this Proposition is a generalization of Lemma 10 in [IMP99] to the case $m > 1$.

Proof of Theorem 4.4: We have to obtain the transversality of $f_k - w_k$ when we restrict $\nabla(f_k - w_k)$ to the distribution defined by $\theta(k)$. Recall that $\theta(k)$ is asymptotically flat, so $\angle_M(D_h, D_k) = O(k^{-1/2})$, where $D_h = \mathbb{C}^n \times \{p\}$ and $D_k = \text{Ker } \theta(k)$. The key idea is that $\nabla(f_k - w_k) = O(1)$, so if $f_k - w_k$ is η -transverse to D_h , then it is, say, 0.9η -transverse to D_k for k large enough. The factor 0.9 can be eliminated by increasing p uniformly. \square

5.2. Proof of Proposition 5.1. Let $f: \mathbb{C}^n \rightarrow \mathbb{C}^m$ be a smooth application. We define the subset $U(f, w, \delta, \sigma)$ of the ball $B(w, \delta)$ of radius δ as the set of points such that f is σ -transverse to w in $\frac{1}{2}B^{2n}$, the ball of radius $1/2$ in \mathbb{C}^n . First we prove the following

Theorem 5.2 (Extension of Theorem 12 in [Do99]). *For any $n, m, \delta > 0$ and $0 < \gamma < 1$ there is a $p = p(n, m, \gamma)$ such that if we define $\sigma = \delta(\log(\delta^{-1}))^{-p}$ then for all the maps $f: B^{2n} \rightarrow \mathbb{C}^m$ verifying that*

$$|f| \leq 1, \quad |\bar{\partial}f| \leq \sigma, \quad |\nabla \bar{\partial}f| \leq \sigma,$$

there exists a connected component of $U(f, w, \delta, \sigma)$ containing another path-connected set $U'(f, w, \delta, \sigma)$ whose volume is at least γ times the total volume of $B(w, \delta)$ and such that given two points $x, y \in U'(f, w, \delta, \sigma)$ is possible to find a smooth curve γ in $U(f, w, \delta, \sigma)$ joining x and y with curvature at each point and length bounded by $\Phi(\delta)$, where Φ is a function depending only on the dimensions n, m .

We repeat the the proof of Theorem 12 of [Do99] taking care of some details.

Proof: The first step in the proof is to approximate f_s by a holomorphic function \hat{f}_s such that $|f_s - \hat{f}_s|_{B, C^1} < c\sigma$ (see Lemma 28 in [Do96]). This process does not hold over all of the unit ball. This is the reason why we restrict ourselves to the ball $B' = \frac{1}{2}B^{2n}$ in the sequel. Then we approximate \hat{f}_s by a polynomial. We can obtain polynomials g_s such that $|g_s - \hat{f}_s|_{B', C^1} \leq c\sigma$ and their degree d can be stimulated by $O(\log(\sigma^{-1}))$.

Adapting notations of [Au97, IMP99] we denote by $Z_{h_s, \epsilon}$ the images of the set of points of B' which are not ϵ -transverse to $\mathbf{0}$ for h_s . We want to prove that one component of the complementary set of $Z_{f_s, \sigma}$ satisfies the required properties. The first observation is that the C^1 -closedness of f_s and g_s assures us that $Z_{f_s, \sigma} \subset Z_{g_s, (c+1)\sigma}$.

We use now the following

Theorem 5.3 (Theorem 26 of [Do99]). *Given a polynomial $g : \mathbb{C}^n \rightarrow \mathbb{C}^m$ of degree d and $\epsilon > 0$ there is a real-algebraic subvariety $A(g) \subset \mathbb{C}^m$ of codimension 2 and degree D such that $Z_{g,\epsilon}$ is contained in the $K\epsilon$ -neighborhood of $A(g)$, where $K, D \leq (d+1)^p$, for some integer p depending only on the dimensions n, m .*

Also we use the following Donaldson's result:

Proposition 5.4 (Proposition 31 in [Do99]). *For each integer N and real number $\theta > 0$, there is a $\mu = \mu(\theta, N)$ with the following property. For any real-algebraic hypersurface $A \subset \mathbb{R}^n$ of degree D and $\epsilon \leq (D+1)^\mu$,*

$$\text{Vol}(B^N \cap A_\epsilon) \leq \theta.$$

Denote $\sigma' = (c+1)\sigma$. With these two results and following the discussion in [Do96] p. 689 about the behaviour of the function $\delta(\log \delta^{-1})^{-p}$ we can assure the $\sigma' = \delta \log(\delta^{-1})^{-p}$ -neighborhood, p a fixed integer, of the bad set $A(g_s)$ has volume arbitrarily small. Also we can assure the same condition for the $3\sigma'$ -neighborhood (changing p slightly). We take a covering of $B(0, \delta)$ by balls $B(x_i, \sigma'/2)$ of centers x_i and radius $\sigma'/2$ and assuring that the covering of balls with radius σ' centered in the same points cover each point of $B(0, \delta)$ only a finite uniform number of times, for instance less than ν times. Denote by \mathcal{C} the set of centers x_i of the balls of the covering contained in the $2\sigma'$ -neighborhood of $A(g_s)$. Recall that the union of these balls is contained in $A_{3\sigma'}(g_s)$, so we can conclude that

$$\sum_{x_i \in \mathcal{C}} \text{vol}(B(x_i, \sigma')) \leq \nu \text{vol}(A_{3\sigma'}(g_s)).$$

And from this expression we can easily obtain a bound for the number of balls in \mathcal{C} as

$$\text{card}(\mathcal{C}) \leq c_u \delta / (\sigma')^2,$$

where c_u is a uniform constant and so this number only depends on δ . But observe that

$$A_{\sigma'}(g_s) \subset \bigcup_{x_i \in \mathcal{C}} B(x_i, \sigma'/2) \subset \bigcup_{x_i \in \mathcal{C}} B(x_i, \sigma') = W_s \subset A_{3\sigma'}(g_s)$$

Recalling that W_s has an arbitrary small volume, a standard isoperimetric inequality gives us that in the complementary of W_s we can find a connected component of arbitrarily big volume (see, for instance, [Au99] for more details). This component will be the set $U'(f, w, \delta, \sigma) = U'$ in the statement of the Theorem. Obviously in the complementary of $A_\sigma(f_s) \subset A_{\sigma'}(g_s)$ there will be a connected component containing U' , this component will be the set $U(f, w, \delta, \sigma) = U$. To finish we have only to check that these two sets satisfy the required properties.

We adapt the ideas of [IMP99]. We call $N = \text{card}(\mathcal{C})$. Observe that we have fixed σ , and then $N = f(\delta)$ and $2\pi\sigma N = g(\delta)$, for some functions f and g . Now take y, z points in the large connected component U of the complementary of Z_s^+ . Denote $L(y, z)$ the straight segment joining them. This segment cuts at most at $2N$ points y_0, z_0, y_1, \dots to the border $Bor = \partial(\bigcup_{x_i \in \mathcal{C}} B(x_i, \sigma'))$. Obviously $L(z_i, y_{i+1}) \subset U$ and $L(y_i, z_i) \subset \bigcup_{x_i \in \mathcal{C}} B(x_i, \sigma')$. We replace the lines $L(y_i, z_i)$ by curves $C(y_i, z_i)$ contained in Bor connecting y_i and z_i . We construct the curves following maximal diameters of the spheres which define the border and so $\text{length}(C(y_i, z_i)) \leq g(\delta)$. Therefore the curve

$$\gamma' = L(y, y_0) \bigcup C(y_0, z_0) \bigcup L(z_0, y_1) \cdots$$

satisfies

$$\begin{aligned} \text{length}(\gamma') &= L(y, y_0) + L(z_0, z) + \sum L(y_i, y_{i+1}) + \sum C(y_i, z_i) \leq 2\delta + \\ &\quad + f(\delta)g(\delta) = \Phi(\delta)/2, \end{aligned}$$

where Φ is some function depending only on the dimensions. Perturbing slightly γ' to make it differentiable and removing it from the border we obtain γ which, bounding enough the perturbation, satisfies $\text{length}(\gamma) \leq \Phi(\delta)$. So the length between two points can be bounded by a function of δ . Moreover, if we translate the diameters in the border Bor till the border of $\bigcup_{x_i \in \mathcal{C}} B(x_i, \sigma'/2)$, we can assure that the curvature of the path can be bounded by $O(1/\sigma')$, again a function of δ .

□

With the precedent result we can easily finish the proof of Proposition 5.1. Recall that f_s satisfies

$$\left| \frac{\partial f_s}{\partial s} \right| \leq 1, \quad \left| \frac{\partial \nabla f_s}{\partial s} \right| \leq 1.$$

This implies by a symple application of the Mean Value Theorem of the Differential Calculus (for vectorial applications) that:

$$|f_s - f_{s+\epsilon}| < \epsilon, \quad |\nabla f_s - \nabla f_{s+\epsilon}| < \epsilon.$$

Recall that given two applications $f, g : \mathbb{C}^n \rightarrow \mathbb{C}^m$ such that $|f - g|_{C^1} < \epsilon$, if f has a left inverse with norm less than η^{-1} , for a large enough uniform constant c_0 , g has a left inverse of norm less than $(\eta - c_0\epsilon)^{-1}$. This implies that

$$(9) \quad U(f_s, 0, \delta, \sigma) \subset U(f_{s+\epsilon}, 0, \delta, \sigma - c_0\epsilon).$$

We choose a number σ , which satisfies the hypothesis of Theorem 5.2, with $\gamma > \frac{1}{2}$. Take an integer q satisfying that

$$(10) \quad \frac{1}{q} < \sigma/c_0.$$

We can choose $x_i \in U(g_{i/q}, 0, \delta, (c+1)\sigma) \cap U(g_{(i+1)/q}, 0, \delta, (c+1)\sigma)$, for $i = 0, \dots, q-1$. Consider the path $H_i = x_i \times [i/q, (i+1)/q]$. There exists a smooth path V_i connecting x_i with x_{i+1} in $U'(g_{(i+1)/q}, 0, \delta, \sigma)$.

Its length is bounded above by $\Phi(\delta)$, as well as its curvature at any point. We parametrize V_i by its arch-length.

We choose a smooth function $\beta: [0, 1] \rightarrow [0, 1]$ satisfying:

$$\beta(x) = \begin{cases} 0, & x \in [0, 1/4], \\ 0 < \beta(x) < 1, & 1/4 < x < 3/4, \\ 1, & x \in [3/4, 1]. \end{cases}$$

and compute $|\beta|_{C_2} = c_b$. Denote $\beta_i(x) = \beta(x) \cdot \text{length}(V_i)$. This function has norm $|\beta_i|_{C_2} < c_b \Phi(\delta)$. Define a path

$$w((i/q + \epsilon) = V_i \left(\beta_i \left(\frac{1}{q} \left(\epsilon + \frac{q}{2} \right) \right) \right), \quad |\epsilon| \leq q/2.$$

The map w is smooth and we obtain $|w|_{C_2} < c_b \Phi(\delta)$, a bound which is a function of δ . Remark that the first derivative depends on the length of the path V_i and the second on its curvature. Finally we observe that $|V_i(s) - w(s)| < \frac{1}{2q}$. It implies, by (9) and (10), that $w(s)$ is $\sigma/2$ transverse to $\mathbf{0}$, for all $s \in [0, 1]$. We can find an integer p' such that $\sigma' = \delta(\log(\delta^{-1}))^{-p'} < \sigma/2$ and the proof is finished.

6. TOPOLOGICAL CONSIDERATIONS.

In this section we will do some simple topological remarks about the contact Lefschetz pencils. In the first subsection we will study the topological relationship between the smooth fibres of a contact fibration.

6.1. Crossing critical curves. Over each of the connected components of $\mathbb{CP}^1 - f_k(\Delta)$ the fibres of f_k are isotopic. Now, we are interested in analyzing the topological behaviour of the fibres when crossing through Δ . We will prove in this subsection the following

Proposition 6.1. *Given a contact fibration (f, Δ, A) , choose a path $\gamma: [0, 1] \rightarrow \mathbb{CP}^1$ such that $\gamma(1/2) \in f(\Delta)$ and the rest of the points of γ are regular values of f . Then $N = f^{-1}(\gamma(1))$ is built up from $N' = f^{-1}(\gamma(0))$ by adding a n -dimensional handle and removing another n -dimensional handle. Therefore $H_j(N) = H_j(N')$ (resp. $\pi_j(N) = \pi_j(N')$) for $j = 0, \dots, n-2$.*

Proof: We restrict ourselves without loss of generality to neighborhoods of $\gamma(1/2)$ and of the critical point of f , where we can define a compatible chart. So, with the usual identifications, we can suppose that $f: \mathbb{R} \times \mathbb{C}^n \rightarrow \mathbb{C}$. Moreover, for simplicity, we will assume that $f(s, z_1, \dots, z_n) = s + z_1^2 + \dots + z_n^2$ (being the general case a straightforward generalization) and the path will be $\gamma(t) = 2(t - 1/2)i$ with a critical value for f at $t = 1/2$.

The proof of the result reduces to show that $B = f^{-1}(\gamma([0, 1]))$ is a cobordism between $N = f^{-1}(\gamma(0))$ and $N' = f^{-1}(\gamma(1))$ with only one surgery of index n . And this follows if we find a Morse function

with a critical point of index n . Choose $h = \text{im}(f) = 2x_1y_1 + 2x_2y_2 + \dots + 2x_ny_n$. We can assume at a neighborhood U of the critical point $\mathbf{0} = (0, \dots, 0)$ that $B \cap U \subset g^{-1}(0)$, where $g(s, x_1, \dots) = \text{Re}(f) = s + x_1^2 + y_1^2 + \dots + x_n^2 + y_n^2$. To compute the index of g we have to restrict ourselves to $\ker \nabla g(\mathbf{0})$ which is

$$\ker \nabla g(\mathbf{0}) = \{(0, z_1, \dots, z_n) : (z_1, \dots, z_n) \in \mathbb{C}^n\}$$

Finally, recall that

$$\nabla \nabla h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2I_n \\ 0 & 2I_n & 0 \end{pmatrix}.$$

Restricting to $\ker \nabla g(\mathbf{0})$ we obtain

$$\nabla \nabla h|_{\ker \nabla g(\mathbf{0})} = \begin{pmatrix} 0 & 2I_n \\ 2I_n & 0 \end{pmatrix},$$

which has index n . □

Now we can obtain a geometrical relationship between the contact submanifolds obtained in Theorem 2.9 as zero sets of transverse sequences with mixed C^3 -bounds. The result will be

Proposition 6.2. *Let S be a line bundle with connection over a closed exact contact manifold (C, D) . Given N_k and N'_k sequences of contact submanifolds obtained as zero sets of transverse sections with mixed C^3 -bounds of the bundles $S \otimes L^{\otimes k}$, then N_k and N'_k are cobordant through a cobordism defined by surgeries of index n .*

Proof: Initially we will suppose that the contact form θ and the almost complex structure J used in the definition and construction of the sequences of sections are coincident in the two cases.

Choose two of such sequences σ_k^1 and σ_k^2 which are η -transverse to $\mathbf{0}$, for some $\eta > 0$. We claim that we can assume without loss of generality that $\sigma_k^1 \oplus \sigma_k^2$ is η' -transverse to $\mathbf{0}$, for some $\eta' > 0$. If we suppose this to be false, we can apply Theorem 2.9 and find a sequence of sections $\tau_k^1 \oplus \tau_k^2$ with mixed C^3 -bounds $(\eta/2, c_R)$, such that they are η' -transverse to $\mathbf{0}$. Define

$$\sigma_{k,t}^j = \sigma_k^j + t\tau_k^j, \quad j = 1, 2.$$

It is easy to check that the sections $\sigma_k^j(t)$ are $\eta/2$ -transverse to $\mathbf{0}$, therefore its zero sets are isotopic. So we can impose directly that $\sigma_k^0 \oplus \sigma_k^1$ is transverse to $\mathbf{0}$.

Define the function

$$\begin{aligned} F_k : C - Z(\sigma_k^1 \oplus \sigma_k^2) &\rightarrow \mathbb{CP}^1 \\ p &\rightarrow \frac{\sigma_k^2(p)}{\sigma_k^1(p)}. \end{aligned}$$

By an analogous argument to that in the precedent lines we assume that ∂F_k is η -transverse to $\mathbf{0}$ away from $Z(\sigma_k^1 \oplus \sigma_k^2)$. Recall that $F_k^{-1}(0) =$

$Z(\sigma_k^1)$ and $F_k^{-1}(\infty) = Z(\sigma_k^2)$. Following the proof of Section 3 we obtain a contact fibration f_k , for k large enough. But, *we do not impose the local model at a neighborhood of A* , therefore we do not perturb any neighborhood of A . We observe that the perturbation at a neighborhood of the set of critical points Δ does not change the counterimages of the points 0 and ∞ . It follows since there exists a ρ_0 -neighborhood, being $\rho_0 > 0$ a uniform constant, of these two fibres which does not intersect the “bad set” Γ , because of the mixed C^3 -bounds of the sections σ_k^1 and σ_k^2 (which assure contactness at a neighborhood of the zero set of the section). So we perturb the sequence F_k in a ρ -neighborhood of Δ , with $0 < \rho < \rho_0$. Therefore, even after performing the needed perturbations, $f_k^{-1}(0) = N_k$ and $f_k^{-1}(\infty) = N'_k$.

By Proposition 6.1 we obtain, following a path between 0 and ∞ , that $f_k^{-1}(0) = N_k$ and $f_k^{-1}(\infty) = N'_k$ are related by a sequence of operations of index n . This finishes the proof if we admit that the complex structures coincide for the two sequences of sections.

Suppose now that the sequence of submanifolds N_k is the zero set of a sequence of sections σ_k with mixed C^3 -bounds respect to an almost complex structure J_0 . Furthermore N'_k comes from s'_k with mixed C^3 -bounds respect to J_1 . Fix a continuous path J_t joining J_0 and J_1 in the moduli of compatible almost complex structures. We use the following

Lemma 6.3. *Let J_0 be a compatible almost complex structure in C . There exists a uniform $\epsilon > 0$ satisfying that for any compatible almost complex structure J such that $|J - J_0| < \epsilon$, there exist two sequences of sections s_k and s'_k with mixed C^3 -bounds, respect to J_0 and J , which are η -transverse to $\mathbf{0}$, for some $\eta > 0$. Moreover the zero sets of s_k and s'_k are isotopic for k large enough.*

Find the uniform constants $\epsilon_t = \epsilon$ provided by Lemma 6.3 for the almost complex structures J_t . This implies, by the continuity of J_t , that there exists $\epsilon'_t > 0$ such that we can find two sequences of sections s_k^t and $s_k^{t'}$ with mixed C^3 -bounds respect to J_t and $J_{t+\epsilon'_t}$ whose zero sets are isotopic, for k large enough.

Now we cover the segment $[0, 1]$ by open segments $(t, t + \epsilon'_t)$. The segment is compact, so we can find a finite subset $(0, \epsilon'_0), (t_1, t_1 + \epsilon'_{t_1}), \dots, (t_N, 1)$ of the precedent set of segments which covers $[0, 1]$. Obviously, without loss of generality, we can choose $t_j + \epsilon'_{t_j} = t_{j+1}$.

Denote $Z(s)$ the zero set of a given section s . Observe that the sets $Z(\sigma_k)$ and $Z(s_k^0)$ are related through a cobordism of index n , for k large enough. But, recall now that $Z(s_k^0)$ and $Z(s_k^{0'})$ are isotopic for k large enough. Again, $Z(s_k^{0'})$ and $Z(s_k^{t_1})$ are related through surgeries of index n ... Following this argument we find that $Z(\sigma_k^0)$ and $Z(\sigma_k^1)$ are related through surgeries of index n .

Finally recall that if s_k is η -transverse to $\mathbf{0}$ with mixed C^r -bounds (c_D, c_R) with respect to a complex structure J and to a contact form θ , then s_k is η/c -transverse to $\mathbf{0}$ with mixed C^r -bounds (cc_D, cc_R) with

respect to the same contact structure J and to a contact form θ' . The constant $c > 0$ only depends on the contact forms θ and θ' . This proves that the result does not depend on the chosen contact form. \square

Proof of Lemma 6.3: Construct a sequence of sections σ_k η -transverse to $\mathbf{0}$ using the globalization argument of Proposition 4.2, for some uniform constant $\eta > 0$. Thus, we obtain

$$\sigma_k = \sum_{j \in J} w_k \cdot s_{k,x_j}^{ref},$$

where the points x_j are elements of the set S with the properties described in the proof of that Proposition. Recall that the definition of σ_k makes sense if we change the almost complex structure J_0 by another one J . The obtained section σ'_k will be different because the reference sections s_{k,x_j}^{ref} depend on the complex structure. However, we claim that σ'_k has mixed C^3 -bounds and that $\sigma_k - \sigma'_k$ has mixed C^3 -bounds $(c_u\epsilon, c'_R)$, where c_u and c'_R are uniform constants. Impose

$$c_u\epsilon < \frac{\eta}{2}.$$

Obviously the constant ϵ can be chosen uniformly and also σ_k and σ'_k have isotopic zero sets. This finishes the proof.

We have only to check that $\sigma_k - \sigma'_k$ have the desired mixed C^3 -bounds. We must follow the proof in [IMP99] to check this condition. We only give the key ideas:

1. Check that $|s_{k,x,J_0}^{ref} - s_{k,x,J}|_{C^2} = O(\epsilon)$.
2. Compute the integer \mathcal{N} in the globalization process for the sequence in J_0 . Check that this constant is enough also for the globalization process with J if we shrink ϵ uniformly in this globalization process. Why uniformly? Because \mathcal{N} is independent of k .

\square

Proposition 6.2 is weaker than the “contact Lefschetz hyperplane theorem” proved in [IMP99], but it is more geometrical and enlightens the behaviour of the generic sections of the bundles $L^{\otimes k}$. It would be interesting to study the different connected components of $\mathbb{CP}^1 - \Delta$ to control the topology of all the “approximately holomorphic” contact zeroes of $L^{\otimes k}$.

7. THE NON-EXACT CASE.

To conclude the discussion we consider now the non-exact case. The important point is the following standard result

Proposition 7.1. *Given a non-exact contact manifold (C, D) . There exists an exact contact manifold (\hat{C}, \hat{D}) which is a non-trivial double covering of C . The projection is a contactomorphism and it can be found a contact form $\hat{\theta}$ in \hat{C} such that the structure \mathbb{Z}_2 -action of the*

covering is a strong anticontactomorphism, i.e. for $\alpha \in \mathbb{Z}_2$, it is $\alpha_*\hat{\theta} = -\hat{\theta}$.

For a simple proof, see [IMP99]. Now, we follow the ideas in [IMP99]. We lift S to the double covering and denote it again by S . It is easy to find an almost complex compatible structure J satisfying that $\alpha_*J = -J$, this implies that $\alpha_*g_k = g_k$. The application α lifts to a morphism of bundles $\tilde{\alpha} : L^{\otimes k} \rightarrow \bar{L}^{\otimes k}$ (in fact recalling from [IMP99] that L is trivialized by construction, we take the identity in each fiber). This morphism preserves the connection. So it is easy to check that if s_k is a sequence of sections of the bundles $S \otimes L^{\otimes k}$ with C^r -bounds (c_D, c_R) with respect to the contact form θ and the almost complex structure J , then α_*s_k is a sequence of sections of the bundle $S \otimes \bar{L}^{\otimes k}$ with respect to the contact form $-\theta$ and to the compatible almost complex structure $-J$ with the same mixed C^r -bounds (c_D, c_R) . But, now we identify the bundle $L^{\otimes k}$ with $\bar{L}^{\otimes k}$ using the anticomplex isomorphism provided by the identity. Then a simple computation shows that α_*s_k is a sequence of sections of the bundles $S \otimes L^{\otimes k}$ with the same mixed C^3 -bounds.

Our only task is to assure that all the objects in the construction are G -invariant. The precedent considerations assure this condition in the local constructions. Now we are going to study the globalization process. The first point is to achieve the set of points S invariantly. Moreover each S_j has to be \mathbb{Z}_2 -invariant. This is only true if the action of \mathbb{Z}_2 is free, because in any other case we would have problems to assure the second property of the set. The perturbation is performed in a similar way that in the standard case. The key idea is that the perturbation term $\tau_{k,x}$ constructed to obtain transversality close to a point x can be transported to the point $\alpha(x)$ by means of $\tau_{k,\alpha(x)} = \alpha_*\tau_{k,x}$. There is no interference between x and $\alpha(x)$, because

$$d_k(x, \alpha(x)) = O(k^{1/2}).$$

The term $\tau_{k,\alpha(x)}$ produces the same transversality as $\tau_{k,x}$ because of the \mathbb{Z}_2 -invariance of the construction.

The perturbation of the function F^{s_k} can be made invariantly. In the set A it is clear. In Δ , the trivialization is invariant, as well as the function H and we can choose w' invariant without loss of generality.

So we have constructed a pencil in \hat{C} which is \mathbb{Z}_2 -invariant. This \mathbb{Z}_2 -invariance property allows us to quotient by the group \mathbb{Z}_2 obtaining new data $A = \hat{A}/\mathbb{Z}_2$, $\Delta = \hat{\Delta}/\mathbb{Z}_2$ and $f = \hat{f}/\mathbb{Z}_2$. It is a trivial exercise to check that the object so defined is a contact pencil on C . \square

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